ON GENERALIZED RANDERS MANIFOLDS

Aly A. Tamim   Nabil L. Youssef
Department of Mathematics, Faculty of Science,
Cairo University, Giza, Egypt.

Dedicated to Professor Radu Miron
on the occasion of his 70th birthday

Introduction

By a Randers' structure on a manifold M we mean a Finsler structure $L^* = L+\alpha$, where L is a Riemannian structure and α is a 1-form on M. This structure was first introduced by Randers [8] from the standpoint of general relativity and was investigated by several authors ([2], [3], [9], ...etc.) from the geometrical viewpoint. Numata [7] studied Randers manifolds in the case where L is a locally Minkowskian structure on M.

In this paper, we replace L by a Finsler structure, calling the resulting manifold a generalized Randers manifold. Such a manifold was studied (using local coordinates) by Matsumoto [3], Tamim [10] and Miron [6]. Our aim is twofold. On one hand, to pursue and develop in depth one of the present authors' study [10] of generalized Randers manifolds. On the other hand, to apply the results obtained in a foregoing paper [12] to generalized Randers manifolds to obtain some new results in that domain. Among many results, we establish a necessary and sufficient condition for a generalized Randers manifold to be a general Landsberg manifold.

It should be noticed that our approach is a global one. That is, it does not make use of the local coordinate techniques (apart from the proof of Theorem 3.1).

1. Notations and Preliminaries

In this section, we give a brief account of the basic concepts necessary for this work. For more details, refer to [1] or [11]. The following notations will be used throughout the paper:

M: a differentiable manifold of finite dimension and of class C^∞.

$\pi_M : TM \longrightarrow M$: the tangent bundle of M.

$\pi : \mathcal{T}M \longrightarrow M$: the subbundle of nonzero vectors tangent to M.

$P : \pi^{-1}(TM) \longrightarrow \mathcal{T}M$: the bundle, with base space $\mathcal{T}M$, induced by π and TM

$\mathfrak{F}(M)$: the $\mathbb{R}$-algebra of differentiable functions on M.

$\mathfrak{X}(M)$: the $\mathfrak{F}(M)$-module of vector fields on M.

$\mathfrak{X}(\pi(M))$: the $\mathfrak{F}(\mathcal{T}M)$-module of differentiable sections of $\pi^{-1}(TM)$.

Elements of $\mathfrak{X}(\pi(M))$ will be called π-vector fields and will be denoted by barred letters $\overline{X}$. Tensor fields on $\pi^{-1}(TM)$ will be called π-tensor fields. The fundamental vector field is the π-vector field $\overline{\eta}$ defined by $\overline{\eta}(u) = (u, u)$ for all $u \in \mathcal{T}M$. The lift to $\pi^{-1}(TM)$ of a vector field X on M is the π-vector field $\overline{X}$ defined by $\overline{X}(u) = (u, X(\pi(u)))$.

The vector bundles $\mathcal{T}M$ and $\pi^{-1}(TM)$ are related by the short exact sequence

$$0 \longrightarrow \pi^{-1}(TM) \xrightarrow{\gamma} T(\mathcal{T}M) \xrightarrow{\rho} \pi^{-1}(TM) \longrightarrow 0,$$

where the vector bundle morphisms are defined by $\rho = (\pi_{\mathcal{T}M}, d\pi)$ and $\gamma(u, v) = j_u(v)$, where j_u is the natural isomorphism $j_u : T_{\pi_M(v)}M \longrightarrow T_u(T_{\pi_M(v)}M)$.

Let ∇ be an affine connection (or simply a connection) in the vector bundle $\pi^{-1}(TM)$. We associate to ∇ the map

$$K : \mathcal{T}M \longrightarrow \pi^{-1}(TM) : X \longmapsto \nabla_X \overline{\eta},$$

called the connection map of ∇. A tangent vector $X \in T_u(\mathcal{T}M)$ is said to be horizontal if $K(X) = 0$. The connection ∇ is said to be regular if

$$T_u(\mathcal{T}M) = V_u(\mathcal{T}M) \oplus H_u(\mathcal{T}M) \qquad \forall u \in \mathcal{T}M,$$

where $V_u(\mathcal{T}M)$ and $H_u(\mathcal{T}M)$ are respectively the vertical and horizontal spaces at u. If M is endowed with a regular connection, we can define a section β of the morphism ρ by $\beta = (\rho\, |_{H(\mathcal{T}M)})^{-1}$. It is clear that $\rho \circ \beta$ is the identity map on $\pi^{-1}(TM)$ and $\beta \circ \rho$ is the identity map on $H(\mathcal{T}M)$. Let $\mathbf{T}$ be the torsion form and $\mathbf{R}$ the curvature transformation of the connection ∇. The horizontal and mixed torsion tensors, denoted respectively by S and T, are defined, for all $\overline{X}, \overline{Y} \in \mathfrak{X}(\pi(M))$, by:

$$S(\overline{X}, \overline{Y}) = \mathbf{T}(\beta\overline{X}, \beta\overline{Y}), \quad T(\overline{X}, \overline{Y}) = \mathbf{T}(\gamma\overline{X}, \beta\overline{Y}).$$

The horizontal, mixed and vertical curvature tensors, denoted respectively by R, P and Q, are defined, for all $\overline{X}, \overline{Y}, \overline{Z} \in \mathfrak{X}(\pi(M))$, by:

$$R(\overline{X}, \overline{Y})\overline{Z} = \mathbf{R}(\beta\overline{X}, \beta\overline{Y})\overline{Z}, \;\; P(\overline{X}, \overline{Y})\overline{Z} = \mathbf{R}(\gamma\overline{X}, \beta\overline{Y})\overline{Z}, \;\; Q(\overline{X}, \overline{Y})\overline{Z} = \mathbf{R}(\gamma\overline{X}, \gamma\overline{Y})\overline{Z}.$$

If $c : I \longrightarrow M$ is a regular curve in M, its canonical lift to $\mathcal{T}M$ is the curve $\tilde{c}$ defined by $\tilde{c} : t \longmapsto dc/dt$. If c is a geodesic in M, we shall denote by $\overline{V}$ the restriction of $\overline{\eta}$ on $\tilde{c}(t)$: $\overline{V} = \overline{\eta}\, |_{\tilde{c}(t)}$.

2. Generalized Randers Manifolds

Let (M, L) be a Finsler manifold. Let g be the Finsler metric associated with (M, L). Using the bundle morphism γ, we define the π-form:

$$\ell = dL \circ \gamma. \tag{1}$$

One can easily show, for all $\overline{X} \in \mathfrak{X}(\pi(M))$, that

$$\ell(\overline{X}) = L^{-1} g(\overline{X}, \overline{\eta}). \tag{2}$$

The angular metric tensor h is defined by:

$$h = g - \ell \otimes \ell. \tag{3}$$

Let ∇ denote the Cartan's connection with respect to g.

Lemma 2.1. *For every* $\overline{X}, \overline{Y}, \overline{Z} \in \mathfrak{X}(\pi(M))$, *we have*

(a) $\nabla_{\gamma\overline{X}} L = \ell(\overline{X})$.

(b) $(\nabla_{\gamma\overline{X}} \ell)(\overline{Y}) = L^{-1} h(\overline{X}, \overline{Y})$.

(c) $(\nabla_{\gamma\overline{X}} h)(\overline{Y}, \overline{Z}) = -L^{-1}\{h(\overline{X}, \overline{Y})\ell(\overline{Z}) + h(\overline{X}, \overline{Z})\ell(\overline{Y})\}$.

(d) $\nabla_{\beta\overline{X}} L = \nabla_{\beta\overline{X}} \ell = \nabla_{\beta\overline{X}} h = 0$.

Let δ be a given 1-form on M. Let $\overline{b}$ be the π-vector field defined in terms of δ by:

$$\delta(X) = g(\overline{b}, \overline{X}) \qquad \forall X \in \mathfrak{X}(T_2M)(M).$$

Writing

$$L^* = L + \alpha, \text{ where } \quad \alpha = g(\overline{b}, \overline{\eta}), \tag{4}$$

L^* defines a new Finsler structure ([10] and [6]) on the manifold M. The Finsler manifold (M, L^*) is called a generalized Randers manifold and (M, L) its associated Finsler manifold.

Using equations (1)–(4), the π-tensors ℓ, h and g associated with (M, L) and the corresponding π-tensors associated with (M, L^*) are related by:

$$\left.\begin{aligned} \ell^* &= \ell + \omega, \text{ where } \omega = d\alpha \circ \gamma \\ h^* &= \tau h, \text{ where } \tau = L^* L^{-1} \\ g^* &= \tau(g - \ell \otimes \ell) + \ell^* \otimes \ell^* \end{aligned}\right\} \tag{5}$$

Proposition 2.2. *Let* $\overline{m}$ *be the* π*-vector field defined by* $\overline{m} = \overline{b} - (\alpha/L^2)\overline{\eta}$, *let* ν *be the* π*-form associated with* $\overline{m}$ *under the duality defined by the metric* g *and let* ϕ *be the* π*-form defined by* $\phi = I - L^{-1}\ell \otimes \overline{\eta}$. *Then, we have*

(a) $\ell(\overline{m}) = 0$.

(b) $\ell^*(\overline{m}) = b^2 - (\alpha/L)^2$, *where* $b^2 = g(\overline{b}, \overline{b})$.

(c) $\nu(\overline{m}) = b^2 - (\alpha/L)^2$.

(d) $\phi(\overline{m}) = \overline{m}$.

(e) $\nu(\overline{X}) = L \nabla_{\gamma\overline{X}} \tau$.

(f) $\phi^* = \phi - L^{*-1} \nu \otimes \overline{\eta}$.

For every $X \in \mathfrak{X}(TM)$ and $\overline{Y} \in \mathfrak{X}(\pi(M))$, let us write

$$\nabla^*{}_X\overline{Y} = \nabla_X\overline{Y} + U(X,\overline{Y}), \tag{6}$$

where U is an $\mathfrak{F}(\mathcal{T}M)$-bilinear mapping $\mathfrak{X}(\mathcal{T}M) \times \mathfrak{X}(\pi(M)) \longrightarrow \mathfrak{X}(\pi(M))$ representing the difference between the two connections ∇^* and ∇.
For every $\overline{X}, \overline{Y} \in \mathfrak{X}(\pi(M))$, we set

$$\left.\begin{array}{rlrl} A(\overline{X},\overline{Y}) &:= U(\gamma\overline{X},\overline{Y}), & B(\overline{X},\overline{Y}) &:= U(\beta\overline{X},\overline{Y}) \\ N(\overline{X}) &:= B(\overline{X},\overline{\eta}), & N_0 &:= N(\overline{\eta}) \end{array}\right\} \tag{7}$$

As a vector field X on $\mathcal{T}M$ can be represented by $X = \gamma KX + \beta\rho X$, it follows from (7) that

$$U(X,\overline{Y}) = A(K(X),\overline{Y}) + B(\rho X,\overline{Y}), \qquad \forall X \in \mathfrak{X}(TM),\ \overline{Y} \in \mathfrak{X}(\pi(M)). \tag{8}$$

From equations (6) and (8), we have

Lemma 2.3. *The two connections ∇ and ∇^* are related by*

$$\nabla^*{}_X\overline{Y} = \nabla_X\overline{Y} + A(KX,\overline{Y}) + B(\rho X,\overline{Y}),$$

for all $X \in \mathfrak{X}(TM)$, $\overline{Y} \in \mathfrak{X}(\pi(M))$.

The π-tensor fields A and B will be determined explicitely later.

One can easily show that

$$\beta^* = \beta - \gamma \circ N. \tag{9}$$

Taking the definition of the torsion tensor T into account and using equations (6), (9) and (5), we get

Proposition 2.4. *For every $\overline{X}, \overline{Y}, \overline{Z} \in \mathfrak{X}(\pi(M))$, we have*

(a) $T^*(\overline{X},\overline{Y}) = T(\overline{X},\overline{Y}) + A(\overline{X},\overline{Y})$.

(b) $T^*(N(\overline{X}),\overline{Y}) - T^*(N(\overline{Y}),\overline{X}) = B(\overline{X},\overline{Y}) - B(\overline{Y},\overline{X})$.

(c) $T^*(\overline{X},\overline{Y},\overline{Z}) = \tau T(\overline{X},\overline{Y},\overline{Z}) + \omega(T(\overline{X},\overline{Y}))\ell^*(\overline{Z}) + A^*(\overline{X},\overline{Y},\overline{Z})$,
where $T(\overline{X},\overline{Y},\overline{Z}) := g(T(\overline{X},\overline{Y}),\overline{Z})$, $T^*(\overline{X},\overline{Y},\overline{Z}) := g^*(T^*(\overline{X},\overline{Y}),\overline{Z})$ and $A^*(\overline{X},\overline{Y},\overline{Z}) := g^*(A(\overline{X},\overline{Y}),\overline{Z})$.

Corollary 2.5. *For all $\overline{X}, \overline{Y}, \overline{Z} \in \mathfrak{X}(\pi(M))$, we have:*

(a) $A^*(\overline{X},\overline{Y},\overline{Z}) = A^*(\overline{X},\overline{Z},\overline{Y}) + \omega(T(\overline{X},\overline{Z}))\ell^*(\overline{Y}) - \omega(T(\overline{X},\overline{Y}))\ell^*(\overline{Z})$.

(a) $A^*(\overline{X},\overline{Y},\overline{\eta}) = -L^*\omega(T(\overline{X},\overline{Y}))$.

Concerning the curvature tensors, we have

Proposition 2.6. *For every $\overline{X}, \overline{Y}, \overline{Z} \in \mathfrak{X}(\pi(M))$, we have*

(a) $$\begin{aligned} R^*(\overline{X},\overline{Y})\overline{Z} + P^*(N(\overline{X}),\overline{Y})\overline{Z} - P^*(N(\overline{Y}),\overline{X})\overline{Z} + Q^*(N(\overline{X}),N(\overline{Y}))\overline{Z} \\ = R(\overline{X},\overline{Y})\overline{Z} + \Omega(\beta\overline{X},\beta\overline{Y})\overline{Z}, \end{aligned}$$
where
$$\begin{aligned} \Omega(\beta\overline{X},\beta\overline{Y})\overline{Z} = {} & (\nabla_{\beta\overline{Y}}B)(\overline{X},\overline{Z}) - (\nabla_{\beta\overline{X}}B)(\overline{Y},\overline{Z}) + A(R(\overline{X},\overline{Y})\overline{\eta},\overline{Z}) \\ & + B(\overline{Y},B(\overline{X},\overline{Z})) - B(\overline{X},B(\overline{Y},\overline{Z})). \end{aligned}$$

(b) $P^*(\overline{X},\overline{Y})\overline{Z}+Q^*(\overline{X},N(\overline{Y}))\overline{Z}=P(\overline{X},\overline{Y})\overline{Z}+\Omega(\gamma\overline{X},\beta\overline{Y})\overline{Z}$,
where
$$\Omega(\gamma\overline{X},\beta\overline{Y})\overline{Z}=-(\nabla_{\gamma\overline{X}}B)(\overline{Y},\overline{Z})+(\nabla_{\beta\overline{Y}}A)(\overline{X},\overline{Z})+A(P(\overline{X},\overline{Y})\overline{\eta},\overline{Z})$$
$$-B(T(\overline{X},\overline{Y}),\overline{Z})+B(\overline{Y},A(\overline{X},\overline{Z}))-A(\overline{X},B(\overline{Y},\overline{Z})).$$

(c) $Q^*(\overline{X},\overline{Y})\overline{Z}=Q(\overline{X},\overline{Y})\overline{Z}+\Omega(\gamma\overline{X},\gamma\overline{Y})\overline{Z}$,
where
$$\Omega(\gamma\overline{X},\gamma\overline{Y})\overline{Z}=(\nabla_{\gamma\overline{Y}}A)(\overline{X},\overline{Z})-(\nabla_{\gamma\overline{X}}A)(\overline{Y},\overline{Z})+A(\overline{Y},A(\overline{X},\overline{Z}))-A(\overline{X},A(\overline{Y},\overline{Z})).$$

Corollary 2.7.

(a) Assume that the π-tensor field B vanishes. Then, $R^*=0$ if, and only if, $R=0$.

(b) The π-tensor field N vanishes if, and only if, N_0 vanishes.

(c) $A(\overline{X},\overline{Y})=A(\overline{Y},\overline{X})$, that is, the π-tensor field A is symmetric.

Proof. (a) follows from Proposition 2.6 (a), taking the fact that $A(\overline{X},\overline{\eta})=0$ into account.
(b) follows from Proposition 2.6(b) and Proposition 2.4(b).
(c) follows from Proposition 2.6 (c). $\Box$

Lemma 2.8. *For all* $\overline{X},\overline{Y}\in\mathfrak{X}(\pi(M))$, *we have*

(a) $(\nabla_{\gamma\overline{X}}\omega)(\overline{Y})=-\omega(T(\overline{X},\overline{Y}))$.

(b) $(\nabla_{\beta\overline{X}}\omega)(\overline{Y})=\ell^*(B(\overline{X},\overline{Y}))+\ell^*(A(N(\overline{X}),\overline{Y}))+L^{-1}h(N(\overline{X}),\overline{Y})-\omega(T(N(\overline{X}),\overline{Y}))$.
In particular, $(\nabla_{\gamma\overline{X}}\omega)(\overline{\eta})=0$ *and* $(\nabla_{\beta\overline{X}}\omega)(\overline{\eta})=\ell^*(N(\overline{X}))$.

Proof. (a) Using Lemma 2.1 and equations (5), we get
$$(\nabla_{\gamma\overline{X}}\ell^*)(\overline{Y})=L^{-1}h(\overline{X},\overline{Y})+(\nabla_{\gamma\overline{X}}\omega)(\overline{Y}).$$
As $\ell(T(\overline{X},\overline{Y}))=0$, Proposition 2.4 and equations (6), (7) and (5) give
$$(\nabla_{\gamma\overline{X}}\ell^*)(\overline{Y})=L^{*-1}h^*(\overline{X},\overline{Y})-\omega(T(\overline{X},\overline{Y})).$$
Taking (5) into account, the result follows from the above two identities.
(b) Using (9), (5), Lemma 2.1 and (a) above, we have
$$(\nabla_{\beta^*\overline{X}}\ell^*)(\overline{Y})=(\nabla_{\beta\overline{X}}\omega)(\overline{Y})-L^{-1}h(N(\overline{X}),\overline{Y})+\omega(T(N(\overline{X}),\overline{Y})).$$
On the other hand, using equations (6), (7) and (9) and Lemma 2.1, we get
$$(\nabla_{\beta^*\overline{X}}\ell^*)(\overline{Y})=\ell^*(A(N(\overline{X}),\overline{Y}))+\ell^*(B(\overline{X},\overline{Y})).$$
The result follows then from the above two equations. $\Box$

The next result gives an explicit expression for the π-tensor field A.

Proposition 2.9. *The π-tensor field A is given by*
$$A=\frac{1}{2L^*}(h\otimes\overline{m}+\nu\otimes\phi+\phi\otimes\nu)-\frac{1}{2{L^*}^2}\{2L^*(\omega\otimes\overline{\eta})T+2\nu\otimes\nu\otimes\overline{\eta}+\nu(\overline{m})h\otimes\overline{\eta}\}.$$

Proof. We first prove, for all $\overline{X},\overline{Y},\overline{Z}\in\mathfrak{X}(\pi(M))$, that

$$A^*(\overline{X},\overline{Y},\overline{Z}) = \frac{1}{2L}\{h(\overline{X},\overline{Y})\nu(\overline{Z})+h(\overline{Y},\overline{Z})\nu(\overline{X})+h(\overline{X},\overline{Z})\nu(\overline{Y})\} \\ -\omega(T(\overline{X},\overline{Y}))\ell^*(\overline{Z}). \tag{10}$$

Using equations (6) and (7), taking the fact that $\nabla^* g^*=0$ into account, we have

$$(\nabla_{\gamma\overline{X}}g^*)(\overline{Y},\overline{Z}) = A^*(\overline{X},\overline{Y},\overline{Z})+A^*(\overline{X},\overline{Z},\overline{Y}).$$

On the other hand, using (5), Lemma 2.1 and Proposition 2.2, we get

$$(\nabla_{\gamma\overline{X}}g^*)(\overline{Y},\overline{Z}) = L^{-1}\{h(\overline{X},\overline{Y})\nu(\overline{Z})+h(\overline{Y},\overline{Z})\nu(\overline{X})+h(\overline{X},\overline{Z})\nu(\overline{Y})\} \\ +\ell^*(\overline{Y})(\nabla_{\gamma\overline{X}}\omega)(\overline{Z})+\ell^*(\overline{Z})(\nabla_{\gamma\overline{X}}\omega)(\overline{Y}).$$

Taking Corollary 2.5 and Lemma 2.8(a) into account, (10) follows from the above two equations.

Now, using equations (5), (3) and (2), taking Proposition 2.2 into account, one can show that:

$$\left.\begin{aligned} g(\overline{m},\overline{Z}) &= \tau^{-1}g^*(\phi^*(\overline{m}),\overline{Z}) \\ h(\overline{X},\overline{Z}) &= \tau^{-1}g^*(\phi^*(\overline{X}),\overline{Z}) \end{aligned}\right\} \tag{11}$$

Substituting (11) into (10), taking (2) into account, it follows from the nondegeneracy of g^* that

$$A(\overline{X},\overline{Y}) = \frac{1}{2L^*}\{h(\overline{X},\overline{Y})\phi^*(\overline{m})+\nu(\overline{X})\phi^*(\overline{Y})+\nu(\overline{Y})\phi^*(\overline{X})\}-\frac{1}{L^*}\omega(T(\overline{X},\overline{Y}))\overline{\eta}.$$

The result follows then from the fact that $\phi^*=\phi-L^{*-1}\nu\otimes\overline{\eta}$. $\square$

Corollary 2.10. *For all* $\overline{X},\overline{Y},\overline{Z}\in\mathfrak{X}(\pi(M))$, *we have*

$$T^*(\overline{X},\overline{Y},\overline{Z}) = \tau T(\overline{X},\overline{Y},\overline{Z})+\frac{1}{2L}\{h(\overline{X},\overline{Y})\nu(\overline{Z})+h(\overline{Y},\overline{Z})\nu(\overline{X})+h(\overline{X},\overline{Z})\nu(\overline{Y})\}.$$

In fact, this formula follows from Proposition 2.4(c) and from equation (10).

Let $(x^i),\ i=1,2,\ldots,n$, be a system of local coordinates on M and let (x^i,y^i) be the associated canonical system of local coordinates on TM and $\mathcal{T}M$. The natural bases of $T_u(\mathcal{T}M)$ and $H_u(\mathcal{T}M)$ are denoted respectively by $(\partial_i,\partial_{,i})_u$ and $(e_i)_u$. The values of the lift $\overline{\partial}_i$ of ∂_i at u form a basis for the fibre over u in $\pi^{-1}(TM)$. We write

$$\nabla_{\partial_i}\overline{\partial}_j=\Gamma^h_{ij}\,\overline{\partial}_h,\quad \nabla_{\partial_{,i}}\overline{\partial}_j=C^h_{ij}\,\overline{\partial}_h,\quad \nabla_{e_i}\overline{\partial}_j=\overline{\Gamma}^h_{ij}\,\overline{\partial}_h.$$

The relations (5) can be expressed locally by:

$$\left.\begin{aligned} {\ell^*}_i &= \ell_i+b_i \\ {h^*}_{ij} &= \tau h_{ij} \\ {g^*}_{ij} &= \tau(g_{ij}-\ell_i\ell_j)+{\ell^*}_i{\ell^*}_j \\ g^{*ij} &= \tau^{-1}g^{ij}+\mu\ell^i\ell^j-\tau^{-2}(\ell^ib^j+\ell^jb^i) \end{aligned}\right\} \tag{12}$$

where $\ell^i = y^i/L$, $\ell_i = g_{ir}\ell^r$, $\mu = (Lb^2+\alpha)/L^*\tau^2$, $b^2 = g_{ij}b^i(x)b^j(x) = b_ib^i$; $b^i(x)$ being the components of the π-vector field $\overline{b}$. We use the notations:

$$\left.\begin{array}{llll} b_{ij} := \nabla_{e_i}b_j, & & b_{i0} & := b_{ik}y^k \\ b_{[ij]} := (b_{ij}-b_{ji})/2, & & b_{(ij)} & := (b_{ij}+b_{ji})/2 \end{array}\right\} \tag{13}$$

After some lengthy but straightforward calculations, using (12) and (13), the π-tensor fields N_0, N and B in (7) are given locally by:

$$\begin{aligned} N^h_0 &= \Gamma^{*h}_{00} - \Gamma^h_{00} \\ &= \ell^{*h}b_{(00)} - 2L^*g^{*hr}b_{[r0]}. \\ N^h_i &= \Gamma^{*h}_{i0} - \Gamma^h_{i0} \\ &= g^{*hk}\left(L^*b_{[ik]} - \ell^*{}_ib_{[k0]}\right) + \ell^{*h}b_{(i0)} + (1/2L^*)h^{*h}_{i}b_{00} + 2L^*g^{*rk}C^{*h}_{ir}b_{[k0]}. \end{aligned}$$

$$\begin{aligned} B^h_{ij} &= \Gamma^{*h}_{ij} - \overline{\Gamma}^h_{ij} - \Gamma^r_{i0}C^{*h}_{jr} \\ &= g^{*hr}\left(\ell^*{}_ib_{[jr]} + \ell^*{}_jb_{[ir]}\right) + \ell^{*h}b_{(ij)} \\ &\quad + (1/2L^*)\left(b_{i0}h^{*h}_{j} + b_{j0}h^{*h}_{i} - g^{*hk}b_{k0}h^*{}_{ij}\right) \\ &\quad + g^{*hp}C^*{}_{ijr}\left\{g^{*rk}\left(L^*b_{[pk]} - \ell^*{}_pb_{[k0]}\right) + \ell^{*r}b_{(p0)} + (1/2L^*)b_{00}h^{*r}_{p}\right. \\ &\quad \left. + 2L^*g^{*mk}C^{*r}_{pm}b_{[k0]}\right\} - g^{*hp}C^*{}_{irp}\left\{g^{*rk}\left(L^*b_{[jk]} - \ell^*{}_jb_{[k0]}\right)\right. \\ &\quad \left. + \ell^{*r}b_{(j0)} + (1/2L^*)b_{00}h^{*r}_{j} + 2L^*g^{*mk}C^{*r}_{jm}b_{[k0]}\right\}, \end{aligned} \tag{14}$$

where $h^r_i = g^{rj}h_{ij}$.

Moreover, Lemma 2.8(b) may be expressed locally by:

$$b_{ij} = B^k_{ij}\ell^*{}_k + A^k_{rj}N^r_i\ell^*{}_k + L^{-1}N^r_ih_{rj} - b_kN^r_iT^k_{rj}. \tag{15}$$

3. Main Results

Let (M, L^*) be a generalized Randers manifold with (M, L) as its associated Finsler manifold.

Using formulae (14) and (15), one can prove

Theorem 3.1. *The π-tensor field B vanishes if, and only if, the ∇-horizontal covariant derivative of ω vanishes* (i.e. $\nabla_{\beta\overline{X}}\omega = 0 \quad \forall\overline{X}$).

Let J be the vector 1-form on $\mathcal{T}M$ defined by $J = \gamma \circ \rho$, then $d_J\alpha = d\alpha \circ J$, where α is the function defined by (4). The proof of the following result is similar to that of Proposition 2 of [11].

Lemma 3.2. *The generalized Randers manifold (M, L^*) and its associated Finsler manifold (M, L) have both the same geodesics if, and only if, $d_J\alpha$ is closed.*

Theorem 3.3. *For the generalized Randers manifold (M, L^*) and its associated Finsler manifold (M, L), the following assertions are equivalent*

(a) *(M,L) and (M,L^*) have the same geodesics.*

(b) *$B(\overline{V},\overline{V})$ vanishes for all $\overline{V}$, where $\overline{V}=\overline{\eta}\mid_{\tilde{c}(t)}$.*

(c) *$d_J\alpha$ is closed.*

(d) *N vanishes identically.*

Proof. (a)$\Longleftrightarrow$ (b): Theorem 2 of [12].
(b)$\Longleftrightarrow$ (c): Lemma 3.2 and Theorem 2 of [12].
(c)$\Longleftrightarrow$ (d): Lemma 3.2, Corollary 2.7(b) and Theorem 2 of [12]. $\square$

Corollary 2.7(a) and Theorem 3.1 imply

Proposition 3.4. *Let $\nabla_{\beta\overline{X}}\omega=0$ for all $\overline{X}\in\mathfrak{X}(\pi(M))$. Then, R^* vanishes if, and only if, R vanishes.*

A Finsler manifold (M,L) is a Berwald manifold [4] if the torsion tensor T satisfies the condition that $\nabla_{\beta\overline{X}}T=0$ for every $\overline{X}\in\mathfrak{X}(\pi(M))$. A Finsler manifold (M,L) is locally Minkowskian [4] if, and only if, $R=0$ and $\nabla_{\beta\overline{X}}T=0$.
Combining Theorems 5 and 6 of [12] and Theorem 3.1, we get

Theorem 3.5. *For the generalized Randers manifold (M,L^*) and its associated Finsler manifold (M,L), suppose that $\nabla_{\beta\overline{X}}\omega=0\quad\forall\overline{X}\in\mathfrak{X}(\pi(M))$.*
Let (M,L) (resp. (M,L^*)) *be a Berwald* (or locally Minkowskian) *manifold. A necessary and sufficient condition for (M,L^*)* (resp. (M,L)) *to be a Berwald* (or locally Minkowskian) *manifold is that $\nabla_{\beta\overline{X}}A=0$ for all $\overline{X}\in\mathfrak{X}(\pi(M))$.*

A Finsler manifold (M,L) is a Landsberg manifold [5] if it satisfies the condition that $P(\overline{X},\overline{Y})\overline{\eta}=0$ for all $\overline{X},\overline{Y}\in\mathfrak{X}(\pi(M))$.
Combining Theorem 7 of [12] and Theorem 3.1, we get

Theorem 3.6. *For the generalized Randers manifold (M,L^*) and its associated Finsler manifold (M,L), suppose that $\nabla_{\beta\overline{X}}\omega=0\quad\forall\overline{X}\in\mathfrak{X}(\pi(M))$.*
(M,L^*) is a Landsberg manifold if, and only if, (M,L) is a Landsberg manifold.

Combining Theorem 1 of [12] and Theorem 3.3, we get

Theorem 3.7. *For the generalized Randers manifold (M,L^*) and its associated Finsler manifold (M,L), suppose that the 1-form $d_J\alpha$ is closed. The horizontal distribution of (M,L^*) is completely integrable if, and only if, the horizontal distribution of (M,L) is completely integrable.*

A general Landsberg manifold [5] is a Finsler manifold such that the trace of the linear map $\overline{Y}\longmapsto P(\overline{X},\overline{Y})\overline{\eta}$ is zero, for all π-vector fields $\overline{X}$. It is characterized by the condition that $\nabla_{\beta\overline{\eta}}C=0$, where C is the π-form obtained from the torsion tensor T by contraction.

Lemma 3.8. *The π-forms C and C^* are related by*

$$C^*=C+\frac{n+1}{2L^*}\,\nu.$$

The proof of this lemma is similar to that found in [10].

Proposition 3.9. *For all π-vector field $\overline{X} \in \mathfrak{X}(\pi(M))$, we have*

$$\begin{aligned}(\nabla^*{}_{\beta^*\overline{\eta}}C^*)(\overline{X}) = (\nabla_{\beta\overline{\eta}}C)(\overline{X}) &- (\nabla_{\gamma N_0}C)(\overline{X}) + C(A(N_0,\overline{X})) \\ &- C(B(\overline{\eta},\overline{X})) + \frac{n+1}{2L^*}\left\{(\nabla_{\beta\overline{\eta}}\nu)(\overline{X}) - (\nabla_{\gamma N_0}\nu)(\overline{X})\right. \\ &\left. + \nu(A(N_0,\overline{X})) - \nu(B(\overline{\eta},\overline{X}))\right\}.\end{aligned}$$

In particular, if the 1*-form $d_J\alpha$ is closed, then*

$$\nabla^*{}_{\beta^*\overline{\eta}}C^* = \nabla_{\beta\overline{\eta}}C + \frac{n+1}{2L^*}\nabla_{\beta\overline{\eta}}\nu.$$

Proof. The first formula follows from Lemma 3.8. The second formula follows from the first one, Theorem 3.3 and from Corollary 2.7(b). □

Now, Proposition 3.9 implies

Theorem 3.10. *For the generalized Randers manifold (M, L^*) and its associated Finsler manifold (M, L), suppose that $d_J\alpha$ is closed.*
Let (M, L) (resp. *(M, L^*)*) *be a general Landsberg manifold. A necessary and sufficient condition for (M, L^*)* (resp. *(M, L)*) *to be a a general Landsberg manifold is that $\nabla_{\beta\overline{\eta}}\nu = 0$.*

Finally, by Theorems 3 and 4 of [12] and Theorem 3.3, we have

Theorem 3.11. *If $d_J\alpha$ is closed, the geodesics of the generalized Randers manifold and those of its associated Finsler manifold have both the same Morse index.*

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